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Characterization of Bipartite Graph and Its Hamiltonicity

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Abstract

In this paper, the characterization of bipartite graph will be first described by using the definitions of cycle, Hamiltonian cycle, odd cycle, and chromatic number in a graph. Then, the conditions for a bipartite graph to be Hamiltonian are investigated. Furthermore, a new necessary condition will be shown for the Hamiltonicity of bipartite graph.

Keywords: Bipartite Graph, Hamiltonian cycle, Hamiltonicity

Introduction

The most useful object in discrete mathematics is a structure called a graph. Graphs were first introduced in the eighteenth century by the Swiss Mathematician Leonard Euler. One of the reasons for the recent interest in graph theory is its applicability in many diverse fields including computer science, chemistry, electrical engineering, and economics. In the study of graph theory, it is necessary to determine the existence of a cycle in which all the vertices appear exactly once except for the starting and ending vertex that appears twice. Such a cycle is called a Hamiltonian cycle and we say a graph is Hamiltonian if it contains a Hamiltonian cycle. Although there are several kinds of graphs, we are interested in the characterization of bipartite graph and its Hamiltonicity.

In this paper, we first describe the basic notations and definitions of graph theory in section 1. Secondly, the characterization of bipartite graph is discussed in section 2. And then, the conditions for a bipartite graph to be Hamiltonian are

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investigated in section 3. In the last section, a new necessary condition for a bipartite graph to be Hamiltonian is exhibited.

1 Terminology and Notations

This section describes some definitions, notations and terminology which will be used throughout this paper. Some special definitions that are only relevant in certain sections will also be given where they are needed. Even worse, the terminology in graph theory is not at all standard, so the reader may find some differences in definitions from any standard textbooks to other (for example, see [1] and [3]).

A ***graph*** G consists of a nonempty finite set V(G) of elements called ***vertices*** and a finite set E(G) ,disjoint from V(G), of unordered pairs of (not necessarily distinct) vertices called ***edges***. We call V(G) the vertex set of G and E(G) the edge set of G . We will often write $G = (V, E)$ which means that V and E are the ***vertex set*** and the ***edge set*** of G , respectively. For convenience, we usually denote the edge $\{u\,v\}$ (where u and v are vertices of G) by uv. If $e = uv$ is an edge of G , then e is said to ***join*** u and v, and these vertices are said to be ***adjacent***. In this case, we also say that e is adjacent to u and v and that u and v are ***endpoints*** of e. A ***loop*** is an edge whose two endpoints are the same. A ***simple graph*** is a graph with no loops and multiple edges. A graph G is called ***complete*** if each vertex in G is connected with every vertex.. A complete graph G on n vertices is denoted by K_n . A ***subgraph*** H of a graph G is a graph H if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph is said to be ***bipartite*** if the vertex set V of G can be partitioned into two subsets X and Y such that each edge connects a vertex in X and a vertex in Y or no edges of G join vertices in the same subset. A ***complete bipartite graph*** is a bipartite graph with vertex set partitioned into a subset X of m elements and a subset Y of n elements such that two vertices are connected by an edge if and only if one is in X and the other is in Y. A complete bipartite graph on m and n vertices is denoted by $K_{m,n}$. The ***degree*** of a vertex v in a graph G is the number of edges incident to it and is denoted by d(v). A ***path*** in a graph G is a sequence of edges that begins at a vertex of a graph and travels along edges of the graph, always connecting pairs of adjacent vertices. The ***length*** of a path is the number of edges contained in it. If u and v are distinct vertices of a graph G and P is a path from u to v, we say that P is a $(u, v)-$***path***. A graph is said to be ***connected*** if there is a path between every pair of its distinct vertices. A ***cycle*** in a graph G is a path that begins and ends at the same vertex (i.e., a closed path of non-zero length that does not contain a repeated edge). A ***Hamiltonian cycle*** is a cycle in a graph G that contains each vertex of G exactly once except for the starting and ending vertex that appears twice. A graph is said to be ***Hamiltonian*** if it contains a Hamiltonian cycle, and otherwise, ***non-Hamiltonian.***

2 Characterization of Bipartite Graph

In this section we begin with the definitions of ***odd cycle*** and ***chromatic number*** in order to describe the characterization of bipartite graph.

2.1 Definition

An odd cycle is a cycle that has an odd number of edges.

2.2 Definition

The length of a shortest $(u, v)-$path is called the distance from u to v and it is denoted by d(u, v).

2.3 Definition

A $t-$coloring of G is a function $C : V \to \{1, 2, \cdots, t\}$ such that $C(u) \neq C(v)$ whenever $uv \in E$.

2.4 Definition

A graph that has a $t-$coloring is said to be $t-$colorable.

2.5 Definition

The minimum value of t for which there exists a $t-$coloring of G is called the chromatic number of G (i.e., the smallest t such that G is $t-$colorable), and it is denoted by $\chi(G)$.

2.6 Theorem

A graph $G = (V, E)$ is bipartite if and only if it contains no odd cycles.

Proof:

Suppose that $G = (V, E)$ is bipartite with bipartition (X,Y).
Let $C = \{v_0 v_1 v_2 \cdots v_k v_0\}$ be any cycle of $G = (V, E)$. We will show that C is even cycle. Without loss of generality we may assume that $v_0 \in X$. Since $G = (V, E)$ is bipartite and $v_0 v_1 \in E$, $v_1 \in Y$. Similarly $v_2 \in X$ and in general $v_{2i} \in X$ and $v_{2i+1} \in Y$. Since $v_0 \in X$, $v_k \in Y$. Then, $k = 2i + 1$, for some i and it follows that C is even cycle. Thus, if $G = (V, E)$ is bipartite, then $G = (V, E)$ contains no odd cycles.
Conversely, suppose that a connected graph $G = (V, E)$ contains no odd cycles.
Let u be any vertex of G and define a partition (X , Y) of V by

$$X = \{x \in V : d(u, x) \text{ is even}\},$$
$$Y = \{y \in V : d(u, y) \text{ is odd}\}.$$

We shall show that (X, Y) is a bipartition of $G = (V, E)$. Suppose that v and w are two vertices of X. Let P be a shortest $(u, v)-$path and Q be a shortest $(u, w)-$path. Denote by u_1 the last vertex common to P and Q. Since P and Q are shortest paths, the $(u, u_1)-$sections of both P and Q are shortest $(u, u_1)-$paths and, therefore, have the same length. Since the lengths of both P and Q are even, the lengths of the $(u_1, v)-$ section P_1 of P and the $(u_1, w)-$section Q_1 of Q must have the same parity. It follows that the $(v, w)-$path $P_1^{-1}Q_1$ is of even length. If v were joined to w, $P_1^{-1}Q_1wv$ would be a cycle of odd length. It contradicts to our assumption. Thus, no two vertices in X are adjacent; no two vertices in Y are adjacent. Hence $G = (V, E)$ is bipartite. ■

2.7 Theorem

A graph $G = (V, E)$ is bipartite if and only if its chromatic number is at most 2.

Proof:

Suppose that $G = (V, E)$ is bipartite with bipartition (X, Y). Then, a coloring of $G = (V, E)$ can be obtained by assigning red to each vertex in X and blue to each vertex in Y. Since there is no edge from any vertex in X to any other vertex in X, and no edge from any vertex in Y to any other vertex in Y, no two adjacent vertices can receive the same color. It follows that the chromatic number of the bipartite graph is at most 2.

Conversely, if we have a $2-$coloring C of a graph $G = (V, E)$, let X be the set of vertices that receive color red and let Y be the set of vertices that receive color blue. Since adjacent vertices must receive different colors, there is no edge between any two vertices in the same partite set. It follows that $G = (V, E)$ is bipartite. ■

3 Investigations on Hamiltonicity of Bipartite Graph

In this section we will investigate the existence of Hamiltonian cycle in simple bipartite graph. Now we have question, "which conditions make a bipartite graph to be Hamiltonian?" The answer must be its number of vertices.

3.1 Definition

The component of a graph is its maximal connected subgraph.

3.2 Proposition

If a graph G has a set of $|S| = k$ vertices whose removal from G results in a graph $G - S$ which has more than k connected components, then G does not have a Hamiltonian cycle.

Proof:

Suppose that G has a Hamiltonian cycle. When leaving vertices of a connected component of $G - S$, a Hamiltonian cycle can go only to S. See Figure 1.

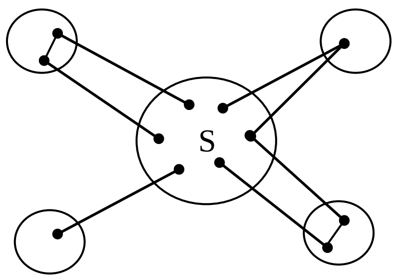

Figure 1

Each arrival in S must be at a different vertex of S. Hence, S must have at least as many vertices as $G - S$ has connected components. It contradicts to the hypothesis of the Proposition. Hence, G does not have a Hamiltonian cycle. ■

3.3 Theorem

If a bipartite graph G with partite sets X and Y is such that $|X| \neq |Y|$, then G is non-Hamiltonian.

Proof:

Suppose without loss of generality that $|X| > |Y|$. Then, removing the vertices of Y from G results in $|X|$ connected components. So, by Proposition 3.2, G does not have a Hamiltonian cycle. Hence, G is non-Hamiltonian. ■

Consequently, we get the following corollary.

3.4 Corollary

Every bipartite graph having odd number of vertices is non-Hamiltonian. ■

3.5 Example

The graph shown in Figure 2 is the Herschel graph. We will investigate whether it is Hamiltonian.

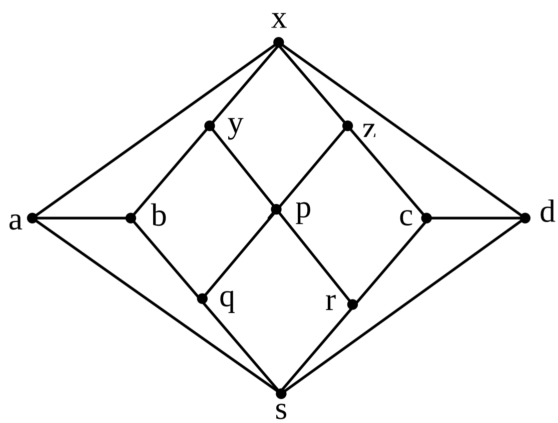

Figure 2: Herschel Graph

We can review the Herschel graph as a bipartite graph with partite sets,

$X = \{d, a, y, z, r, q\}$ and

$Y = \{x, s, c, b, p\}$, as shown in Figure 3.

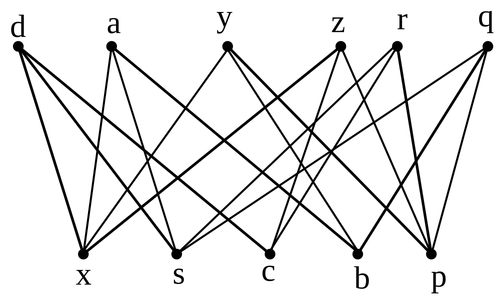

Figure 3

Since $|X| = 6$ and $|Y| = 5$, $|X| \neq |Y|$.

By Theorem 3.3, the Herschel graph is non-Hamiltonian.

Consequently, one should consider whether the bipartite graph is Hamiltonian if $|X| = |Y|$.

The following counterexample shows that although a bipartite graph with partite sets X and Y has $|X| = |Y|$, it is non-Hamiltonian.

Consider the graph shown in Figure 4.

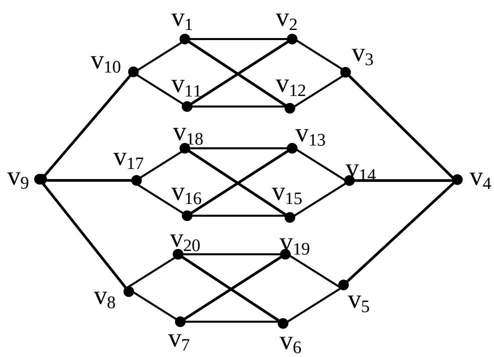

Figure 4

If the vertices v_4 and v_9 in above graph are removed, then the graph $G - \{v_4, v_9\}$ has three connected components. Thus, by Proposition 3.2, the graph does not have Hamiltonian cycle.

The graph of Figure 4 can be transformed into a bipartite graph with | X | = | Y |, where

$X = \{v_1, v_3, v_5, v_7, v_9, v_{11}, v_{14}, v_{16}, v_{18}, v_{20}\}$ and

$Y = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{13}, v_{15}, v_{17}, v_{19}\}$,

as shown in Figure 5.

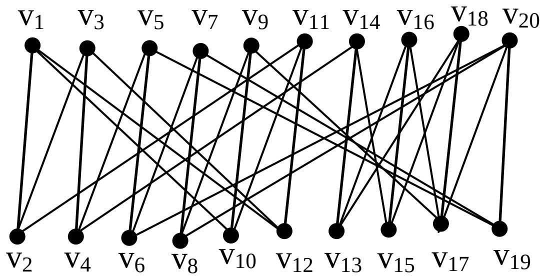

Figure 5

Since | X | = 10 and | Y | = 10, the redrawing bipartite graph shown above has two equal partite sets.

It is non-Hamiltonian since there does not exist a Hamiltonian cycle. ■

4 Necessary Condition for a Bipartite Graph to be Hamiltonian

Our investigations from the previous section are facing with the conflict of which conditions make a bipartite graph to be Hamiltonian. So, the purpose of this section is to show a result we produced which is an agreeable condition for a bipartite graph to be Hamiltonian.

4.1 Theorem

A complete bipartite graph $K_{m,n}$ is Hamiltonian if and only if $m = n$, for all m, n ≥ 2.

Proof:

Suppose that a complete bipartite graph $K_{m,n}$ is Hamiltonian.

Then, it must have a Hamiltonian cycle which visits the two partite sets alternately. Therefore, there can be no such cycle unless the two partite sets have the same number of vertices. If m = n = 1, it is clear that $K_{m,n}$ contains no Hamiltonian cycle.

Thus, we get m = n, for all m, n ≥ 2.
Conversely, suppose that a bipartite graph with partite sets,

$$X = \{x_1, x_2, \cdots, x_m\} \text{ and}$$
$$Y = \{y_1, y_2, \cdots, y_n\},$$

is complete and its two partite sets have the same number of vertices (i.e., $m = n$). Then, it is obvious that $x_1 y_1 x_2 y_2 \cdots x_m y_n x_1$ is a Hamiltonian cycle. Hence $K_{m,n}$ is Hamiltonian. ■

The following example demonstrates Theorem 4.1.

4.2 Example

Consider the bipartite graphs shown in Figures 6 and 7.

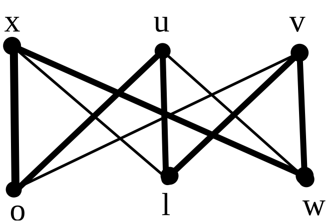

Figure 6

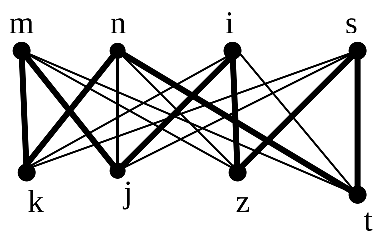

Figure 7

The graph of Figure 6 is a complete bipartite graph $K_{3,3}$ with m = n = 3. It satisfies the conditions of Theorem 4.1. One of its Hamiltonian cycles is

x l v w u o x.

The graph of Figure 7 is a complete bipartite graph $K_{4,4}$ with m = n = 4. It satisfies the conditions of Theorem 4.1. One of its Hamiltonian cycles is

m j i z s t n k m.

Again, in the following Figure 8, although it is a complete bipartite graph, $K_{2,3}$, it is non-Hamiltonian since the numbers of vertices in each partite set are different.

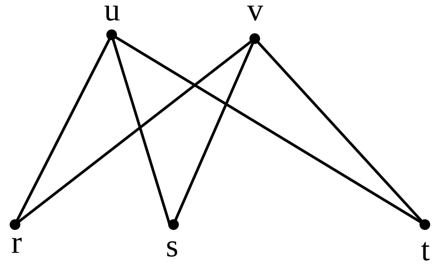

Figure 8

■

Conclusion

The problem of finding necessary and sufficient conditions for a graph to be Hamiltonian is both interesting and difficult. In this paper, we presented our investigations on the existence of Hamiltonian cycles in bipartite graph. We had exhibited a new necessary condition for a bipartite graph to be Hamiltonian. We firmly believe that there remains a lot to determine in this line of research. Furthermore, one may analyze the Hamiltonicity of bipartite digraphs and the number of Hamiltonian cycles in bipartite digraphs.

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